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Note

Graphs with many large degrees ☆

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Abstract

Graphs with many large degrees have subgraphs with all degrees large.

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1. Introduction

The greedy algorithm is a standard proof technique for embedding a 'small' graph into a 'large' host-graph G. It works especially well when all degrees in G are large. If we only know that G has many edges, then often the following fact is used (it is part of graph theory folklore):

Lemma 1. *Every non-empty graph G has a subgraph H such that $\delta(H) > t(G)/2$ ($t(G)$ denotes the average degree of G).*

We will use the following version.

Lemma 2. *Every non-empty graph G has a subgraph H such that*

$$t(H) \geqslant t(G) \quad \text{and} \quad \delta(H) > \tfrac{1}{2}\, t(H).$$

Such a subgraph can be obtained by successively deleting vertices with degrees not exceeding half the average degree of the current graph.

We prove an analogue of Lemma 2 for graphs with many large degrees. Specifically, the average degree of G in the lemma is replaced by the median degree of G, or some other quantile of the degree sequence of G.

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For example, assume that the only information about a given graph G is that half of its vertices have degrees at least k. Using Lemma 2 will only guarantee a subgraph H of G with $\delta(H) > k/4$, but using the new method will provide a subgraph H' with $\delta(H') > k/3$. This was used to prove the Loebl-Komlós-Sós conjecture for graphs with girth at least 7 [2].

1.1. Notation

For basic graph concepts see the monograph of Bollobás [1].

We write $V(G)$ and $E(G)$ for the vertex and edge sets of the graph G. The *order* of G is $v(G) = |V(G)|$, and the *size* of G is $e(G) = |E(G)|$. For a graph G and a subset U of its vertices, $G[U]$ is the restriction of G to U (i.e., the subgraph induced by U).

We write (v) (or $_G(v)$) for the degree of v (in G), and $\delta(G)$, $\Delta(G)$, and $t(G)$ for the minimum, maximum, and average degrees in G. The *median degree*, $\mu(G)$, of a graph G is the largest integer k such that at least half the vertices of G have degrees at least k. In general, for $0 \leqslant q < 1$, $\mu_q(G)$ is the *q-quantile* of the degree sequence of G, that is, $\mu_q(G)$ is the largest integer k such that at least a $(1-q)$-proportion of the vertices of G have degree k or more. (Thus $\mu(G) = \mu_{1/2}(G)$.) In other words, if $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n$ is the degree sequence of G, then $\mu_q(G) = d_{[(1-q)n]}$.

Given a graph G and a positive integer λ, let $L(G) = L_\lambda(G) = \{v \in V(G) : (v) \geqslant \lambda\}$ (large vertices), $S(G) = S_\lambda(G) = V(G)(G)$ (small vertices), and $G_\lambda = G[L_\lambda]$.

The *maximum minimal degree of* G is defined as $MMD(G) = \max\{\delta(H) : H \subset G\}$, and the *maximum average degree of* G is $MAD(G) = \max\{t(H) : H \subset G\}$. Furthermore, let

$$f(k,n) = \max\{e(G) : v(G) = n,\ MMD(G) < k\},$$
$$g(k,n,\lambda) = \max\{|L_\lambda(G)| : v(G) = n,\ MMD(G) < k\},$$
$$h(x,n,\lambda) = \max\{|L_\lambda(G)| : v(G) = n,\ MAD(G) < x\}.$$

2. The theorems

In the following theorem, we determine the function g exactly. Since $g(k,n,\lambda) = n$ if $\lambda < k$, and it is 0 if $n < \lambda + 1$, we will assume that neither of these extremes occurs.

Theorem 3 (Main theorem). *Let $k,n,\lambda \in \mathbf{N}$, $\lambda \geqslant k-1$, and $n \geqslant \lambda + 1$. Then,*

$$(k,n,\lambda) = \min\left\{\left\lfloor \frac{(n-k)(k-1)}{\lambda - k + 1} \right\rfloor,\ n - (\lambda - k + 1)\right\}.$$

(*The right-hand side is defined to be $n - (\lambda - k + 1) = n$ for $\lambda = k - 1$.*)

This will imply the following analogue of Lemma 1.

Theorem 4. *Every non-empty graph G has a subgraph H with $\delta(H) > \mu(G)/3$.*

Theorem 5. *Every graph G satisfies*

$$MAD(G) \geqslant \tfrac{2}{3}\,\mu(G).$$

That is, every graph G has a subgraph H with $t(H) \geqslant (2/3)\mu(G)$. In general, for $0 \leqslant q < 1$,

$$MAD(G) \geqslant \tfrac{2-2q}{2-q}\,\mu_q(G),$$

and hence, if G is non-empty then

$$MMD(G) > \frac{1-q}{2-q}\,\mu_q(G).$$

Combining Theorem 5 and Lemma 2, we get the following strengthening of Theorem 4.

Theorem 6. *Every non-empty graph G has a subgraph H such that*

$$t(H) \geqslant \frac{2}{3}\mu(G) \quad \textit{and} \quad \delta(H) > \frac{1}{2}\,t(H).$$

3. The extremal graphs

First we define *sliding window graphs*. Let $k, n \in bfN$, $n \geqslant k-1$, and let the sliding window graph $W(k,n)$ be defined as (V,E), where $V = \{1, 2, \ldots, n\}$, and two different vertices i and j are connected if $|i-j| < k$. The crucial property of sliding window graphs is that $W(k,n)$ does not contain a subgraph with minimum degree at least k. Equivalently, recursively deleting all vertices with degrees less than k will result in an empty graph.

It is easy to see from the definition that the maximum degree of $W = W(k,n)$ is $\Delta = \min\{2k-2, n-1\}$, and the number of edges of W is

$$e(W(k,n)) = (n - k/2)(k-1).$$

In fact, $W(k,n)$ has two vertices of degree $k-1$, two vertices of degree k, two vertices of degree $k+1, \ldots$, two vertices of degree $\Delta - 1$, and the leftover $n - 2(\Delta - k + 1) \geqslant 1$ vertices all have degree Δ.

Given positive integers k, ℓ, λ with $\ell \geqslant k-1$, $\lambda \geqslant k-1$, and a non-negative integer s, we define the class of $LS(k,\ell,s,\lambda) - graphs$ as follows. We start with the window graph $W = W(k,\ell)$ on a set L of size ℓ (large vertices), add a set S of s new vertices (small vertices), and then connect some vertices in L to some vertices in S in such a way that in the obtained graph every vertex in L has degree at least λ and every vertex in S has degree at most $k-1$.

Such graphs only exist for certain values of the parameters. The following two conditions are necessary and sufficient. For a vertex $x \in L$, define the deficiency of x as $\mathrm{def}(x) = \max\{\lambda - {}_W(x), 0\}$. Thus, we need at least $\mathrm{deficit}(k,\ell,\lambda) := \sum_{x \in L} \mathrm{def}(x)$ edges between L and S. But the number of edges between them can be at most $(k-1)|S|$. Hence we need

$$(k-1)s \geqslant \text{deficit}(k,\ell,\lambda). \tag{1}$$

Also, we clearly need

$$s \geqslant \max\{\text{def}(x) : x \in L\} = \lambda - k + 1. \tag{2}$$

The sufficiency of these conditions can be seen by going through all vertices in L one by one and connecting them to the necessary number of vertices in S by using the vertices of S in a cyclic fashion.

Hence, as a lower bound for $g(k,n,\lambda)$, we want to find the largest integer ℓ satisfying

$$n - \ell \geqslant \max\left\{\frac{\text{deficit}(k,\ell,\lambda)}{k-1},\ \lambda - k + 1\right\}, \tag{3}$$

which is equivalent to

$$\ell \leqslant \min\left\{\left\lfloor n - \frac{\text{deficit}(k,\ell,\lambda)}{k-1}\right\rfloor,\ n - (\lambda - k + 1)\right\}. \tag{4}$$

A simple computation shows that

$$\text{deficit}(k,\ell,\lambda) = \begin{cases} (\lambda-k+1)(\lambda-k+2) & \text{if } k-1 \leqslant \lambda \leqslant \Delta, \\ n\lambda - 2e(W) = n(\lambda-2k+2) + k(k-1) & \text{if } \lambda \geqslant \Delta \end{cases}$$

and our condition becomes

$$\ell \leqslant \min\left\{\left\lfloor \frac{(n-k)(k-1)}{\lambda-k+1}\right\rfloor,\ n - (\lambda - k + 1)\right\}.$$

The right-hand side here is at least $k-1$, so it is an appropriate choice for ℓ in $W(k,\ell)$. Thus, we obtained the following construction.

Theorem 7. *Given positive integers k,ℓ,λ, $\lambda \geqslant k-1$, $n \geqslant \lambda+1$, there is an n-graph G without subgraphs of minimum degree at least k such that*

$$L_\lambda(G) \geqslant \left\{\left\lfloor \frac{(n-k)(k-1)}{\lambda-k+1}\right\rfloor,\ n - (\lambda - k + 1)\right\}.$$

Consequently,

$$g(k,n,\lambda) \geqslant \left\{\left\lfloor \frac{(n-k)(k-1)}{\lambda-k+1}\right\rfloor,\ n - (\lambda - k + 1)\right\}.$$

4. Proofs of Theorem 3 and Theorem 4

For the proof of Theorem 3, we need the following lemmas.

The first one (see e.g. [1, Exercise IV/11]) is a strengthening of Lemma 2.

Lemma 8. *Let* $k, n \in \mathbf{N}$, $n \geqslant k - 1$. *Then*

$$f(k,n) = (n - k/2)(k - 1).$$

This immediately implies an upper bound on the function g via the next lemma.

Lemma 9. *Let* $k, n, \lambda \in \mathbf{N}$, $\lambda \geqslant k$, *and assume that G has no subgraph with minimum degree at least k. Then*

$$|L_\lambda(G)| \leqslant \frac{e(G)}{\lambda - k + 1} \left(\leqslant \frac{f(k,n)}{\lambda - k + 1} \right).$$

Consequently,

$$g(k,n,\lambda) \leqslant \frac{f(k,n)}{\lambda - k + 1}.$$

Proof of Lemma 9. Let G be an n-graph without a subgraph of minimum degree at least k ($MMD(G) < k$). We are going to order the vertices of G the way it is done by the usual greedy coloring algorithms. We write $G_1 = G$, and define the sequence G_i of graphs as follows. For $i \geqslant 1$, let v_i be a vertex of minimum degree in G_i and let $G_{i+1} = G_i - v_i$. Let S be the set of edges with right endpoint (under this ordering) in L_λ. Then, since from every vertex there are at most $k - 1$ edges going to the right, any vertex in L_λ is the right endpoint for at least $\lambda - k + 1$ edges. Thus,

$$|L_\lambda|(\lambda - k + 1) \leqslant |S| \leqslant e(G). \qquad \square$$

Proof of Theorem 3. Theorem 7 provides a lower bound, so we only need a matching upper bound. Let G be an n-graph without a subgraph of minimum degree at least k ($MMD(G) < k$). We need to show that

$$|L_\lambda(G)| \leqslant \min \left\{ \left\lfloor \frac{(n-k)(k-1)}{\lambda - k + 1} \right\rfloor,\ n - (\lambda - k + 1) \right\}. \tag{5}$$

Since the right-hand side of (5) is at least $k - 1$, we are done if $|L_\lambda| \leqslant k - 1$. Assume now that $|L_\lambda| \geqslant k$. By the definition of f, we have $e(G_\lambda) \leqslant f(k, |L_\lambda|)$ and $e(G) \leqslant f(k,n)$. Thus,

$$f(k,n) \geqslant e(G) \geqslant |L_\lambda|\lambda - e(G_\lambda) \geqslant |L_\lambda|\lambda - f(k, |L_\lambda|).$$

Hence, by Lemma 8 (which can be applied since $n \geqslant |L_\lambda| \geqslant k$),

$$(n - k/2)(k - 1) \geqslant |L_\lambda|\lambda - (|L_\lambda| - k/2)(k - 1)$$

implying

$$|L_\lambda| \leqslant \left\lfloor \frac{(n-k)(k-1)}{\lambda - k + 1} \right\rfloor.$$

On the other hand, G_λ must have at least one vertex of degree at most $k - 1$, which then must have at least $\lambda - (k - 1)$ neighbours outside G_λ. Thus, $|S_\lambda| \geqslant \lambda - k + 1$, and so $|L_\lambda| \leqslant n-(\lambda-k+1)$. Hence,

$$|L_\lambda| \leqslant \min\left\{\left\lfloor\frac{(n-k)(k-1)}{\lambda-k+1}\right\rfloor,\ n-(\lambda-k+1)\right\}. \qquad \square$$

Proof of Theorem 4. (*We show how the statement follows directly from Theorem 3. It is obtained later again as the special case $q=\frac{1}{2}$ in Theorem 5.*). *Let G be a non-empty graph on n vertices with $\mu(G)=\mu$, and let $k=\lfloor 1+\mu/3\rfloor$—the least integer greater than $\mu/3$. We want to show that G has a subgraph of minimum degree k or more. Indeed, if it did not, then we could apply Theorem 3 with $\lambda=\mu$ and would have*

$$\frac{n}{2} \leqslant |L_\mu| \leqslant g(k,n,\mu) \leqslant \frac{(n-k)(k-1)}{\mu-k+1} \leqslant \frac{n-k}{2} < \frac{n}{2},$$

which is impossible. (The conditions $n \geqslant \lambda+1=\mu+1$ and $\mu \geqslant k$ automatically hold whenever $\mu \geqslant 1$, which we may assume.) Thus $MMD(G) \geqslant k$, which completes the proof. $\square$

5. Proof of Theorem 5

Lemma 10. *Let $n,\lambda \in \mathbf{N}$, let G be an n-graph and write $w=t(G_\lambda)$. Then*

$$t(G) \geqslant \frac{|L_\lambda|}{n}(2\lambda-w).$$

Proof. The lemma follows from the inequality

$$e(G) \geqslant |L_\lambda|\lambda - e(G_\lambda) = |L_\lambda|(\lambda - w/2)$$

and the identity $t(G)=2e(G)/n$. $\square$

We will use Lemma 10 to prove the following estimate on the function h.

Theorem 11. *Let $n,\lambda \in \mathbf{N}$, let $x \leqslant \lambda$ be a positive real number, and let G be a graph on n vertices. If*

$$|L_\lambda(G)| \geqslant \frac{x}{2\lambda-x}n$$

then $MAD(G) \geqslant x$, that is, G has a subgraph H with $t(H) \geqslant x$. Consequently,

$$h(x,n,\lambda) < \frac{x}{2\lambda-x}n.$$

Proof of Theorem 11. Let $w=t(G_\lambda)$. If $w \geqslant x$, choose $H=G_\lambda$ and we are done. If $w<x$, choose $H=G$. We need to show that $t(H) \geqslant x$. Indeed, using Lemma 10 we get

$$t(H)=t(G) \geqslant \frac{|L_\lambda|}{n}(2\lambda-w) > \frac{|L_\lambda|}{n}(2\lambda-x) \geqslant x. \quad \square$$

Proof of Theorem 5. Apply Theorem 11 with $\lambda = \mu_q(G)$ and $x = \mu_q(G)(2 - 2q)/(2 - q)$. □

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References

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